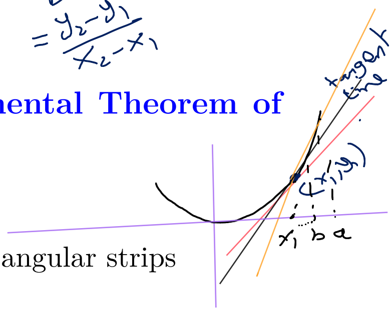


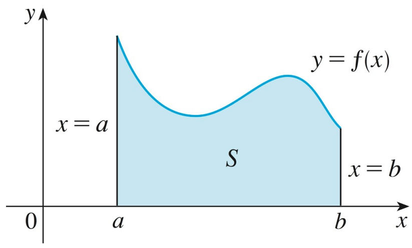
Sections 5.1-5.3: Area, the Definite Integral, and Fundamental Theorem of Calculus

Objective: In this lesson, you learn how to

- ☐ Find the area under a curve by estimating the sum of areas of rectangular strips
- ☐ Define a definite integral as the limit of a Riemann sum.
- ☐ Apply the Fundamental Theorem of Calculus to find areas under curves and definite integrals.

I. The Area Problem

We attempt to find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the $x-$axis.

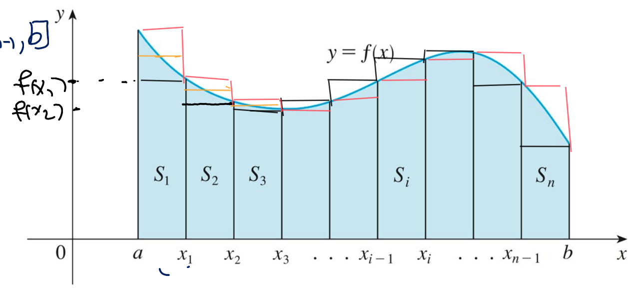


To this end, first approximate the region under the curve from a to b by rectangles and then take the **limit** of the areas of these rectangles as the number of rectangles **increases**.

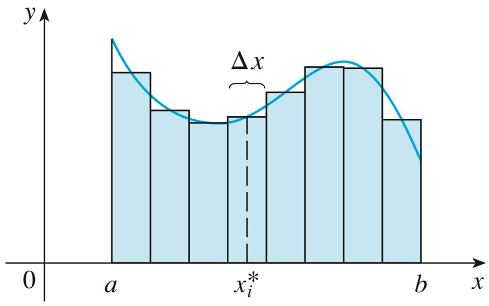
Steps:

1. Divide the interval $[a, b]$ into n subintervals of equal length $\triangle x = (b - a)/n$.

$[a,b] \longrightarrow [a,x_1],[x_1,x_2], \ldots ,[x_{n-1},b]$



2. Next, pick n arbitrary points $x_1^*, x_2^*, ..., x_n^*$, called **sample points** (**representative points**), from the first, second, . . . , and n^{th} subintervals, respectively.



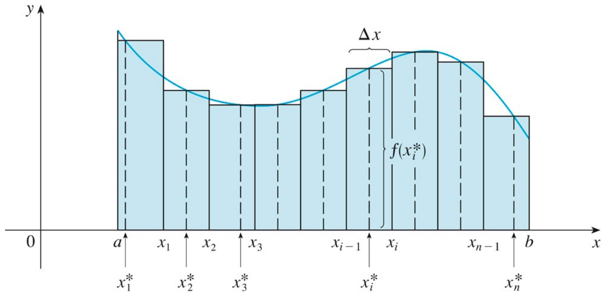
3. The area of the i^{th} rectangular strip whose width is Δx and whose height $f(x_i^*)$ is

$$f(x_i^*)\,\Delta x.$$

So the sum of the areas of these rectangular strips is

$$R_n \approx f(x_1^*)\triangle x + f(x_2^*)\triangle x + ... + f(x_n^*)\triangle x = \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

The sum on the right-hand side of this expression is called a **Riemann sum**.



The Area under the Graph of a Function

Let f be a nonnegative continuous function on $[a,b]$. Then, the area (R) of the region S under the graph of f is

$$R = \lim_{n\to\infty}[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)]\triangle x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $x_1^*, x_2^*, ..., x_n^*$ are arbitrary points in the n subintervals of $[a,b]$ of equal width $\triangle x = (b-a)/n$.

We can choose the **sample points** to be the

- Right endpoints:

$$x_i^* = a + i\triangle x$$

- left endpoints:

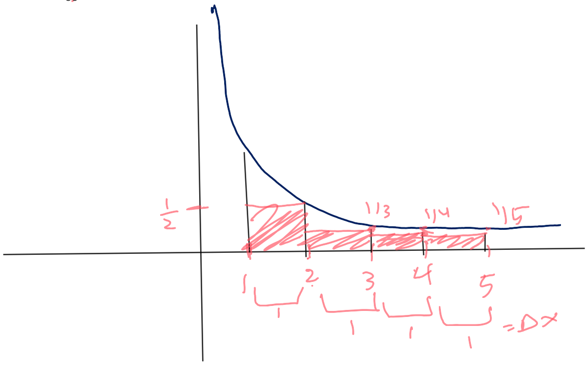
$$x_i^* = a + (i-1)\triangle x$$

- Mid-points:

$$x_i^* = \frac{1}{2}(x_{i-1} + x_i) = \textbf{ midpoint of } [x_{i-1}, x_i]$$

Example 1: Find the area under the graph of $f(x) = \dfrac{1}{x}$ from $\overset{a\ \ b}{[1, 5]}$ by using $\overset{n}{\underline{4}}$-rectangles.

$n = 4, \quad a = 1, \quad b = 5$



$\boxed{1}$ $\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{4} = 1$

$\boxed{2}$ split $[1,5]$ into 4-subinterval

1 2 3 4 5

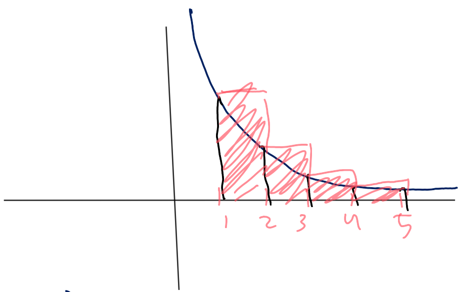
$[1,2], [2,3], [3,4], [4,5]$

$\boxed{3}$ Right-end points are $\overset{x_1^*}{2}, \overset{x_2^*}{3}, \overset{x_3^*}{4}, \overset{x_4^*}{5}$

$f(2) = \frac{1}{2}, \; f(3) = \frac{1}{3}, \; f(4) = \frac{1}{4}, \; f(5) = 1/5$

$$R_4 = \sum_{i=1}^{4} \Delta x \, f(x_i^*)$$
$$= 1 \cdot \left(f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) \right)$$
$$= f(2) + f(3) + f(4) + f(5)$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.2833$$

under estimate of the actual Area.



$\boxed{4}$ left-end points are $\overset{x_1^*}{1}, \overset{x_2^*}{2}, \overset{x_3^*}{3}, \overset{x_4^*}{4}$

$$L_4 = \sum_{i=1}^{4} \Delta x \, f(x_i^*)$$
$$= 1 \cdot \left(f(1) + f(2) + f(3) + f(4) \right)$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cong 1.579603$$

over estimate of the actual Area

$1.2833 < \text{Actual area} < 1.5$

II. The Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, x_1, x_2, $\ldots$, $x_n = b$ be the endpoints of the subintervals and we let x_1^*, x_2^*, $\ldots$, x_n^* be any sample points in the subintervals; so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$.

The **definite integral** of f from a to b is defined as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

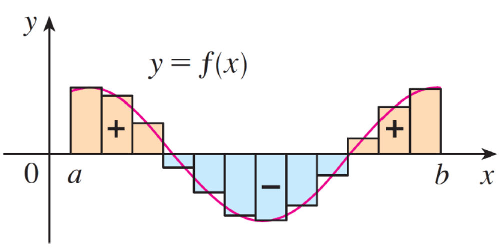
provided that this limit exists and gives **the same value** for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

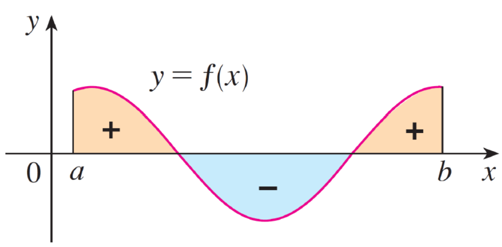
There are several facts to note about definite integrals:

- The symbol $\int$ was introduced by **Leibniz** and is called an **integral sign.**
- In the notation $\int_a^b f(x)\,dx$ (which is all one symbol), $f(x)$ is called the **integrand**, a and b are called the **limits of integration**, a is the **lower limit**, and b is the **upper limit**.
- The procedure of calculating an integral is called **integration**.
- The definite integral $\int_a^b f(x)\,dx$ is a **number** and **does not** depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

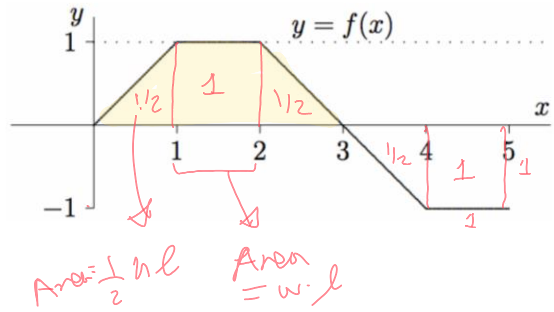
$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

- **Geometric Interpretation of $\int_a^b f(x)dx$:**
  - If f is nonnegative and continuous on $[a, b]$. Then the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b
  - If f is continuous on $[a, b]$. Then the definite integral can be interpreted as a **net area**, that is, the area of the region above the $x-$axis and below the graph of f, minus the area of the region below the $x-$axis and above the graph of f.





Example 2: Consider the graph of $f(x)$, find



a. Find the value of the integral $\int_0^3 f(x)\,dx$

$$\int_0^3 f(x)\,dx = \frac{1}{2}(1)(1) + (1)(1) + \frac{1}{2}(1)(1)$$
$$= 2.$$

b. Find the value of the integral $\int_0^5 f(x)\,dx$

$$\int_0^5 f(x)\,dx = \underbrace{2}_{\text{area above the x-axis}} - \underbrace{\frac{3}{2}}_{\text{area below the x-axis}} = 1/2$$

Example 3: Evaluate $\int_0^2 \sqrt{4 - x^2}\,dx$

The area of a circle is $= \pi r^2$

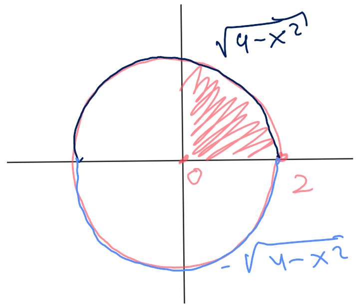
$y = \sqrt{4-x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4 = 2^2$

eqn. of a circle $c = (0,0)$, $r = 2$

$$\int_0^2 \sqrt{4-x^2}\,dx$$
$$= \frac{1}{4}\pi\cdot(r)^2$$
$$= \frac{\pi}{4}\cdot(2)^2$$
$$= \pi$$



eqn. of a circle is

$(x-h)^2 + (y-k)^2 = r^2$

$c = (h, k)$

radius $= r$

x	y
0	-3
3	0

Example 4: Evaluate $\int_1^4 (x-3)\,dx$

$x^2 + y^2 = 4$

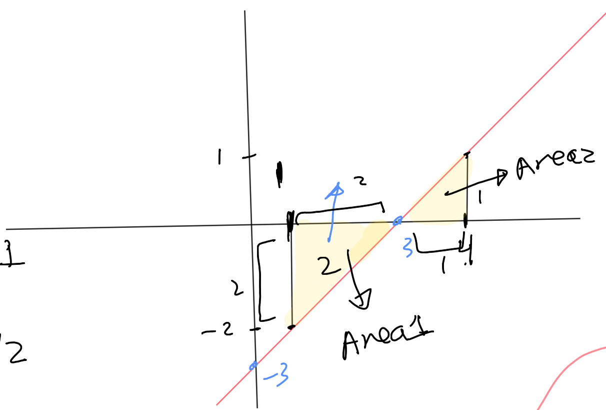
$y^2 = 4 - x^2$

$y = \pm\sqrt{4 - x^2}$

Area 1 $= \frac{1}{2}\cdot 2\cdot 2 = 2$

Area 2 $= \frac{1}{2}\cdot 1\cdot 1 = 1/2$

$$\int_1^4 (x-3)\,dx = \text{Area } 2 - \text{Area } 1$$
$$= \frac{1}{2} - 2 = -3/2$$



$\sqrt{9} = +3$

$x^2 - 9 = 0$

$\sqrt{x^2} = \sqrt{9}$

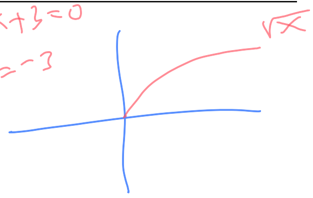
$|x| = 3$

$x = 3$ or $x = -3$

$(x-3)(x+3) = 0$

$x - 3 = 0$ or $x + 3 = 0$

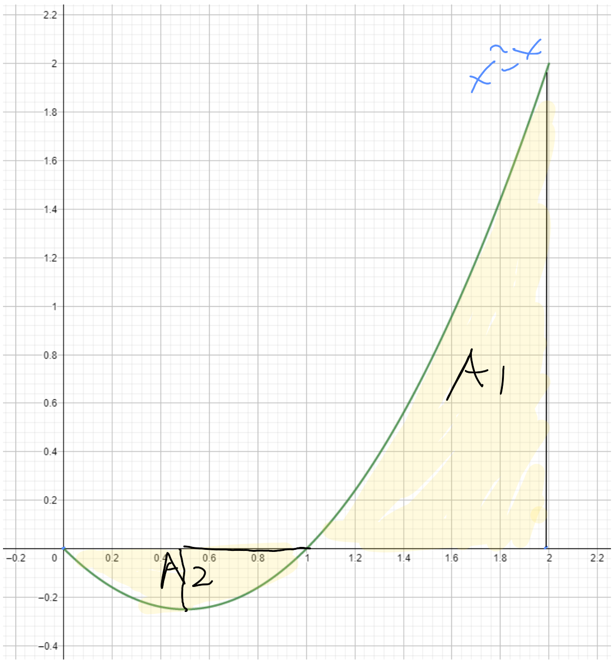
$x = 3$ or $x = -3$



Example 5: Evaluate $\int_0^2 (x^2 - x)\, dx$

$$\int_0^2 x^2 - x\, dx$$

$$= A_1 - A_2$$



It is not easy to find the area of A_1 and A_2.

we can find the areas by using Riemann sum. But, it is a lot of work.

How do we find this?

Using FTC section 5.3.

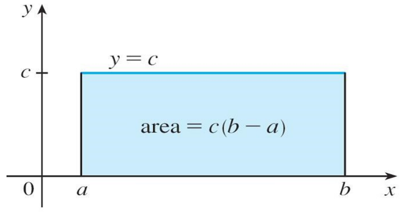
III. Properties of the Definite Integral

1. In the notation $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But if $b < a$, then $\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$. So
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

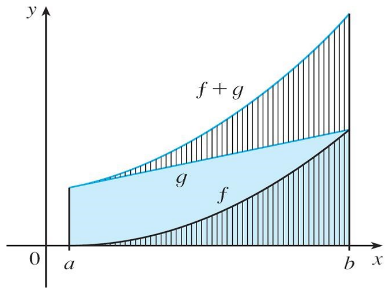
If $\int_2^3 f(x)\,dx = 5$ then $\int_3^2 f(x)\,dx = -5$

2. If $a = b$, then $\Delta x = 0$, so $\int_a^b f(x)\,dx = 0$. $\int_2^2 f(x)\,dx = 0$

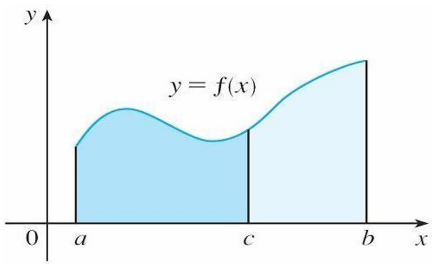
3. If $f(x) = c$ is a constant function, then
$$\int_a^b c\,dx = c\,(b-a).$$
The graph of f forms a rectangle with base length $b - a$ and height c if $c > 0$ and $a < b$.



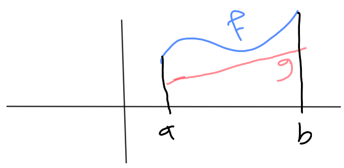
4. For positive functions, the area under $f + g$ is the area under f plus the area under g.
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$



5. The area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b. This is easily seen if $f(x) \geq 0$ for $a < c < b$, but a general case is not as easy.
$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$



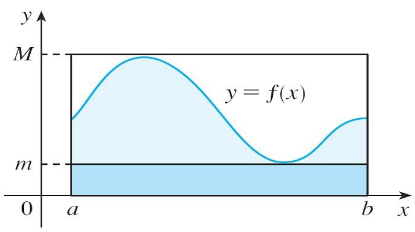
While the properties discussed so far are true if $a \leq b$ or $b \leq a$, the next three properties called **Comparison Properties** of the Integral are true only if $a \leq b$.



6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a).$$



IV. The Fundamental Theorem of Calculus

The *Fundamental Theorem of Calculus* establishes a connection between the two branches of calculus: **differential calculus** and **integral calculus.**

Differential calculus arose from the **tangent problem** and integral calculus arose from the **area problem**. Newton and Leibniz used the relationship between differentiation and integration as inverse processes to develop calculus into a systematic mathematical method

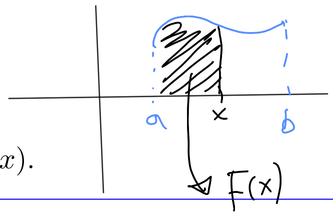
$(\sqrt[x]{x})^{x}$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$. Then, the function F defined by

$$F(x) = \int_a^x f(t)\, dt \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $F'(x) = f(x)$.



$\frac{d}{dx}\int_a^x$

The **second part** of the Fundamental Theorem of Calculus, which follows easily from the **first part**, provides us with a much simpler method for the evaluation of integrals.

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$. Then,

$$\int_a^b f(t)\, dt = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

In applying the Fundamental Theorem of Calculus, it is convenient to use the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

Example 6: Let S be the region under the graph of $f(x) = \frac{1}{x}$ on the interval $[1, 5]$. Use the Fundamental Theorem of Calculus to find the area of S. i.e. Find

$$\int_1^5 \frac{1}{x}\, dx$$

$f(x) = \frac{1}{x} \longrightarrow F(x) = \text{Ln}|x|$

$\int_1^5 \frac{1}{x}\, dx = \text{Ln}|x| \Big|_1^5 = \text{Ln}(5) - \cancel{\ln(1)}^{\,0}$

$= \text{Ln}\, 5$

Example 7: Evaluate

a. $\int_0^2 x^2 - x\,dx$

$f(x) = x^2 - x \longrightarrow F(x) = \frac{x^3}{3} - \frac{x^2}{2} + C$

$\int_0^2 x^2 - x\,dx = \frac{x^3}{3} - \frac{x^2}{2} + c \Big|_0^2 = \left(\frac{2^3}{3} - \frac{2^2}{2} + c\right) - \left(\cancel{\frac{0^3}{3}} - \cancel{\frac{0^2}{2}} + c\right)$

$= \frac{8}{3} - 2 + \cancel{c} - \cancel{c}$

b. $\int_1^3 (3x^2 + e^x)\,dx$

$\int_1^3 3x^2 + e^x\,dx = \frac{\cancel{3}x^3}{\cancel{3}} + e^x \Big|_1^3$

$= (3^3 + e^3) - (1^3 + e^1)$

$= 8 + e^3 - e$

c. $\int_0^4 x(x^2 - 1)\,dx = \int_0^4 x^3 - x\,dx$

$= \frac{x^4}{4} - \frac{x^2}{2} \Big|_0^4$

$= \left(\frac{4^4}{4} - \frac{4^2}{2}\right) - (0)$

#